Supplemental Materials for "Mean-field Approximations for Coupled Populations of Generalized Linear Model Spiking Neurons with Markov Refractoriness" by Taro Toyoizumi, Kamiar Rahnama Rad, and Liam Paninski, Neural Computation, Volume 21, Issue 5, pp. 1203-1243.

Color figures:

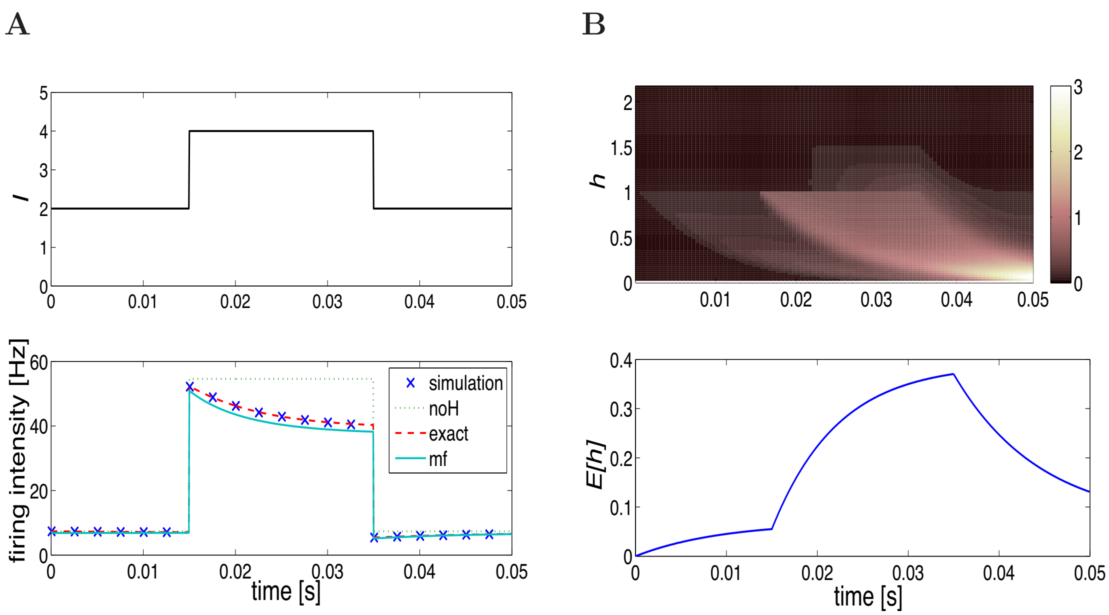

Figure 2: Direct solution of the master equation in the one-dimensional case. (A) *Top:* A single neuron is simulated with a pulse input $I(t)$ shown. *Bottom:* Comparison of the firing rates computed via: direct Monte Carlo sampling (simulation); the inhomogeneous Poisson case, with no history dependence, i.e., $w = 0$ (noH); the master equation of equation. 3.2 (exact); and the mean-field approximation methods in the limit of $N_c \to \infty$ introduced in section 4. the mean-field approximation gives good approximations of the output firing intensity. (B) *Top:* The probability distribution $P(h, t)$; color axis indicates the height of the probability density. The self-history term $h(t)$ was the convolution of the output spike train and a single exponential filter with time constant $\tau = 10$ ms and weight $J = -1$; thus the neuron was inhibited for a brief period following each spike. Note the discontinuous behavior visible at stimulus onset, where probability mass jumps significantly towards $h = 1$, then decays back down towards zero, according to equation 3.1. *Bottom:* The mean of h calculated from the probability distribution $P(h, t)$. This $E[h]$ also increases following the input pulse, which has the effect of slowing down the firing rate: since the weight J is negative, the history term is inhibitory here.

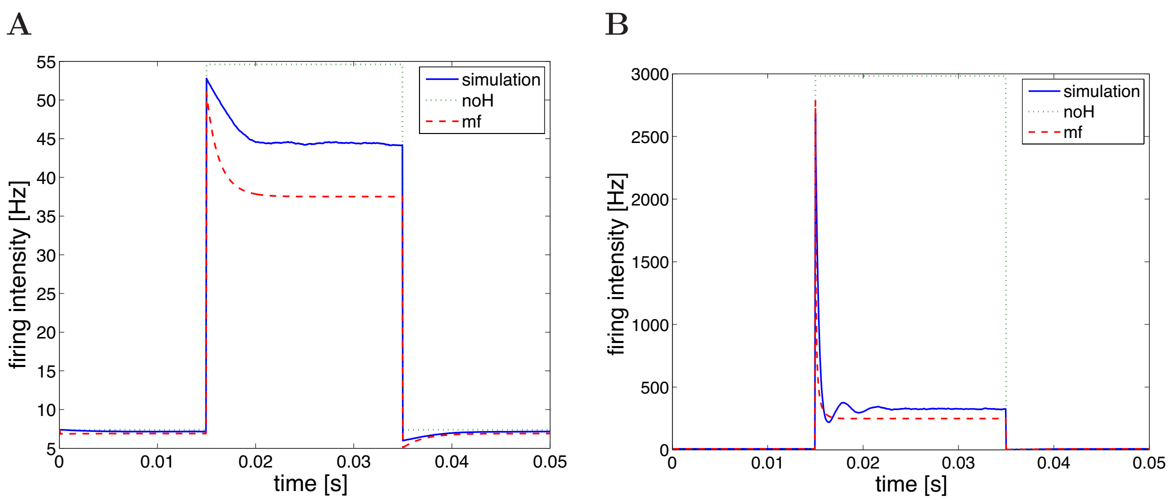

Figure 3: Comparison of the direct Monte Carlo simulation of a single neuron (solid), the firing intensity without recurrent input H (dotted), and the mean-field approximation (dashed). (A) Low step input (baseline $I = 2$, peak $I = 4$) is applied. The self-inhibition term is set to $J = -5$ and $\tau = 2$ ms. (B) High step input (baseline $I = 2$, peak $I = 8$) is applied. The self inhibition terms is set to $J = -5$ and $\tau = 2$ ms. Because of the large non-Gaussian fluctuations of input caused by the strong self-inhibition, the mean-field approximation loses accuracy here.

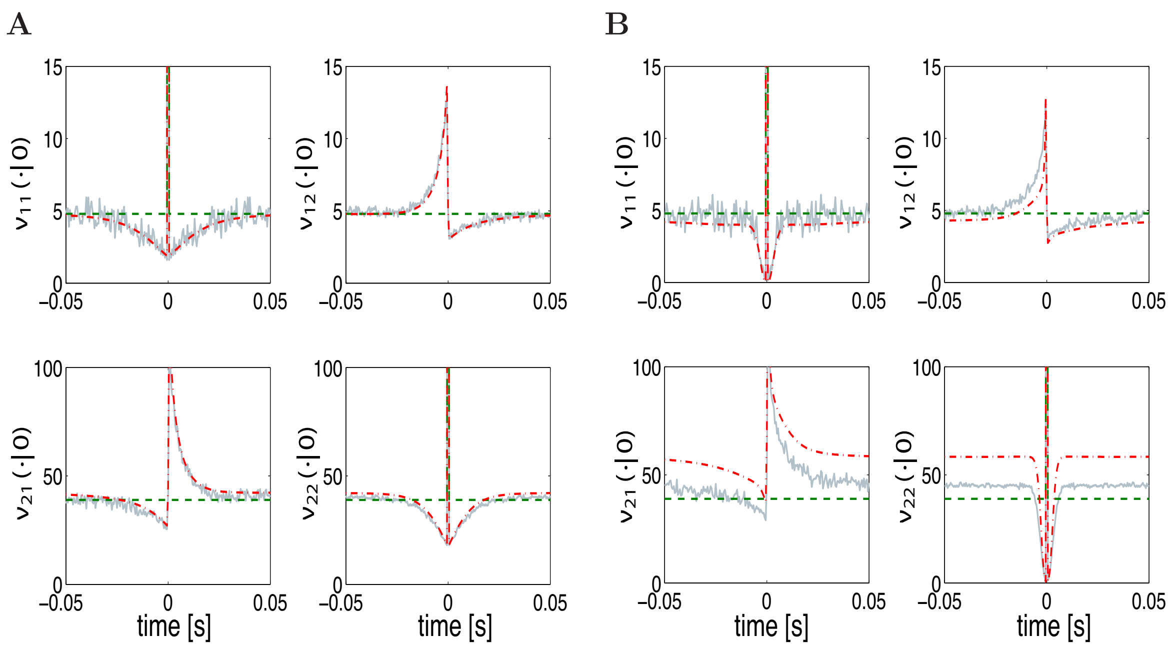

Figure 4: Comparison of cross-correlation functions of two neurons calculated by direct simulation (solid), calculated from the mean-field approximation (see equations 4.9 and 4.10; dashed), and the mean-field approximation with finite size correction (see equations 4.11 and 4.12; dot-dashed). (A) Weak self-interaction: parameters are set to $J_{11} = -1, J_{22} = -1, J_{21} = 1, J_{12} = -0.5$, $\tau_{11} = 10\text{ms}, \tau_{22} = 10\text{ms}, \tau_{21} = 10\text{ms}, \tau_{12} = 20\text{ms}$, and $I_1 = 2, I_2 = 4$. The result with finite size correction is obtained after 5 iterations of the self-consistent equations. The calculation in the mean-field limit captures only the steady state firing intensity but not any temporal cross-correlation function. Cross-correlation functions are well approximated with the finite size correction terms. (B) Strong self-interaction: parameters are set to $J_{11} = -5, J_{22} = -5, J_{21} = 1, J_{12} = -0.5$, $\tau_{11} = 2\text{ms}, \tau_{22} = 2\text{ms}, \tau_{21} = 10\text{ms}, \tau_{12} = 20\text{ms}$, and $I_1 = 2, I_2 = 4$. Because of the large non-Gaussian fluctuation of the input caused by the strong self-inhibition term, the mean-field approximations does not work very well. In this case the finite size solution converges rather slowly. Hence, the result is after 10 iterations of the self-consistent equations.

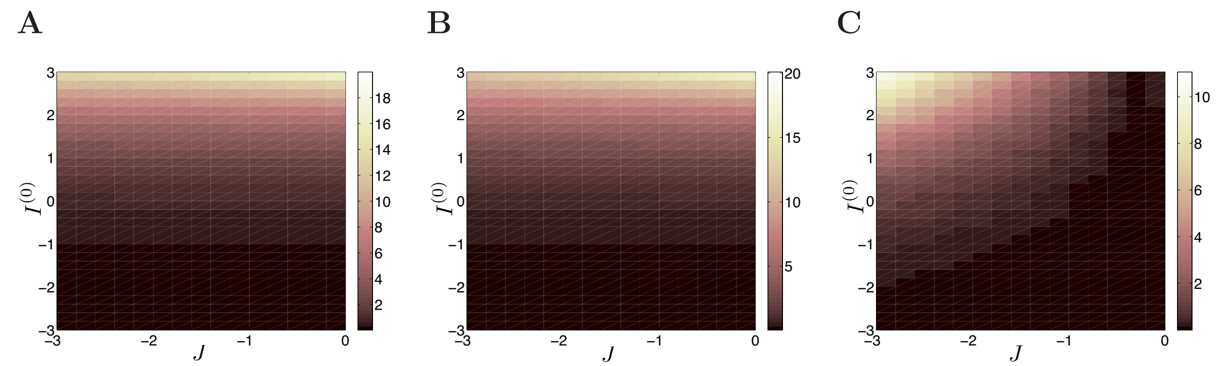

Figure 5: Comparison of (A) a simulated firing rates of a single neuron and (B) the self-consistent solutions of equation 4.13 for different input levels, $I^{(0)}$, and self-inhibition strengths, J. (C) The percentage errors of the mean-field approximation are shown with respect to the simulated firing intensity. We set $\tau = 10$ ms.

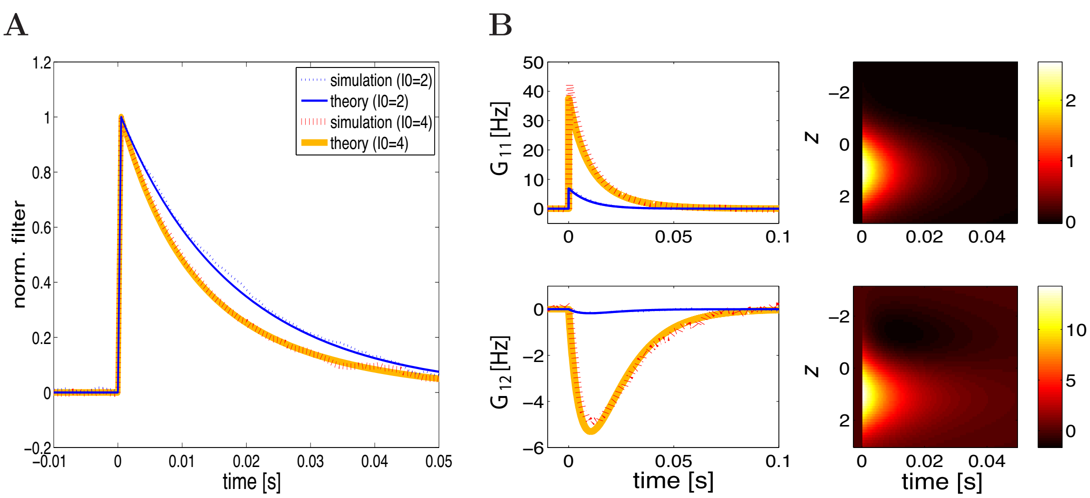

Figure 6: Comparison of the analytically derived and simulated input-output filters. (A) A simulated normalized linear input-output filter for $I^{(0)} = 2$ (thin-dotted) and $I^{(0)} = 4$ (thick-dotted) are compared with analytical results at $I^{(0)} = 2$ (thin-solid) and $I^{(0)} = 4$ (thick-solid) of equation 4.18. Because the strength of the self-inhibition component changes with the baseline input through $f'(I^{(0)} + \mu^{(0)})$, the input-filter is sharpened for high baseline input. Other parameters are set to $J = 1$, $\tau = 10$ ms, $K(t) = e^{-t/\tau_I}\Theta(t)$ with $\tau_I = 20$ ms and $\sigma_\xi = 1$ Hz. All filter amplitudes are normalized by their maximum amplitudes. (B) Left panels show the comparison of analytically derived input-output filters $G_{11}(t)$ and $G_{12}(t)$ of equation 4.20 (solid lines) and simulated input-output filters (dotted lines) for $I^{(0)} = 2$ (thin lines) and $I^{(0)} = 4$ (thick lines). Right panels show neuron 1's spatio-temporal input-output filter $\tilde{G}_1(t, z)$ for $I^{(0)} = 2$ (top) and $I^{(0)} = 4$ (bottom). See text for parameter values.

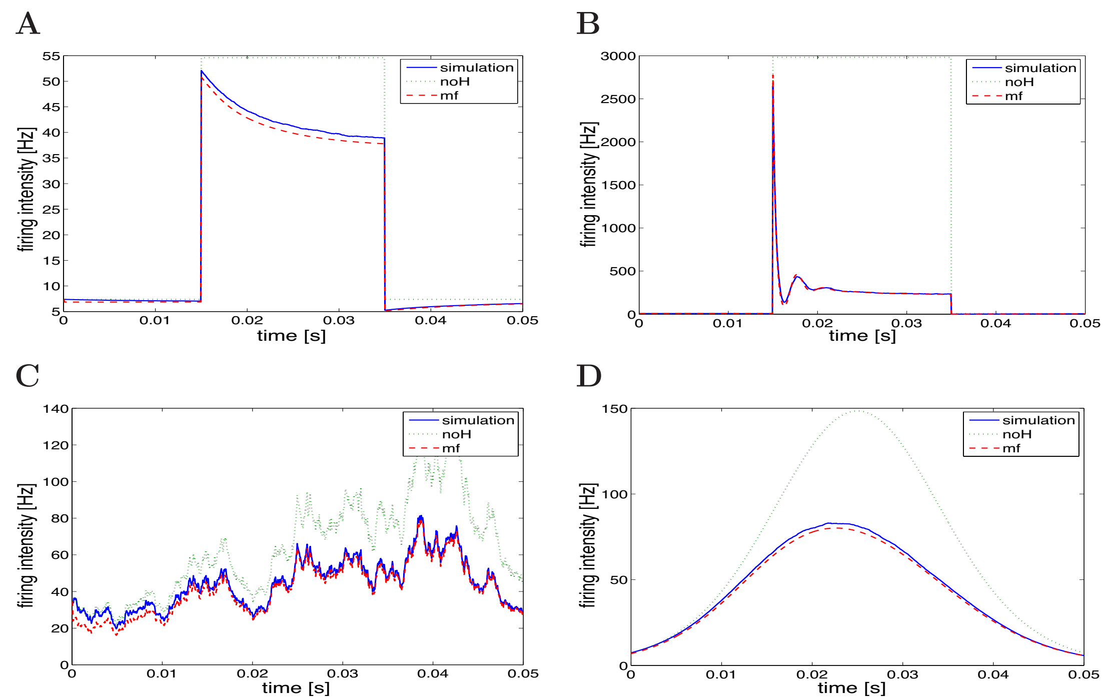

Figure 7: Comparison of the mean firing intensity obtained from direct simulation of a single neuron (solid), without recurrent input H and no Markov refractory effect (dotted), and the mean-field equation in the limit of $N_c \to \infty$ (dashed). In all the cases, the markov refractory effect with $\tau_r = 2\text{ms}/(M-1)$ was included. (A) Weak step input (baseline $I = 2$, peak $I = 4$) and refractoriness $M = 3$; $J = -1$, $\tau = 10$ ms. Although the model includes strong Markov refractory effect, the mean-field approximation is not abolished. (B) Strong step input (baseline $I = 2$, peak $I = 8$) and refractoriness $M = 10$; $J = -1$, $\tau = 10$ms. The mean-field approximation also approximates very well the transient oscillation caused by the the strong refractory effect and steep current. (C) The mean-field approximation works for colored noise input: $\tau_I \frac{dI(t)}{dt} = -I(t) + \xi(t)$ with $E[\xi(t)] = 4$, $Cov[\xi(t), \xi(t')] = 0.1\delta(t-t')$, and $\tau_I = 20$ ms; $M = 3$, $J = -1$, $\tau = 10$ms. The initial error is due to the preset initial values deviate from the true ones. (D) Sinusoidal input input: $I(t) = 2 + \sin(2\pi \cdot 10\text{Hz} \cdot t)$; $M = 3$, $J = -1$, $\tau = 10$ms. The mean-field results give good approximations.

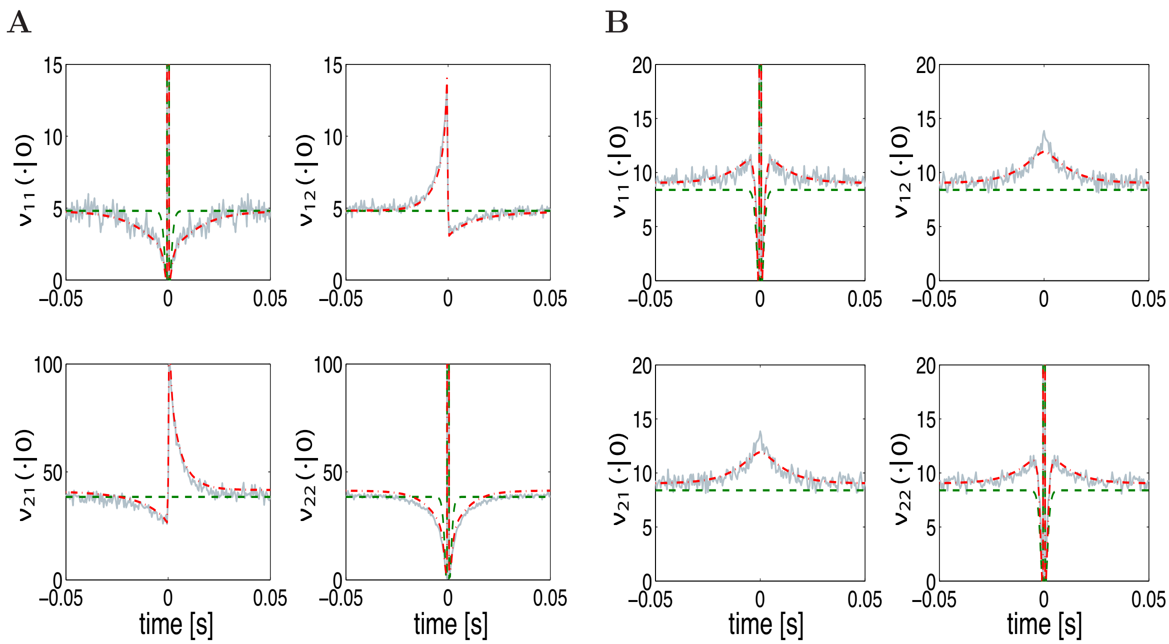

Figure 8: Comparison of spike cross-correlation functions of two neurons obtained by direct numerical simulation (solid), mean-field approximation in the limit of $N_c \to \infty$ (dashed), and with finite size correction terms (dot-dashed). While the approximation in the limit of $N_c \to \infty$ only captures the strong Markov refractory effect, the weaker refractory effect caused by the self-interaction terms is well captured with the finite size correction. The finite size correction provides a good approximation despite the strong refractoriness because of an explicit evaluation of the Markov refractory effect. Mean-field equations are iteratively solved over 5 iterations. (A) Two neurons are directly coupled: parameters are set to $J_{11} = -1, J_{22} = -1, J_{12} = -0.5, J_{21} = 1,\ \tau_{11} = 10\text{ms}, \tau_{22} = 10\text{ms}, \tau_{12} = 20\text{ms}, \tau_{21} = 10\text{ms}$, $M = 3,\ I_1 = 2, I_2 = 4$. (B) Two neurons are not directly connected but receive common input from a third neuron: parameters are set to $J_{13} = J_{23} = 2$, $\tau_{13} = \tau_{23} = 10\text{ms}$, $M = 3$, $I_1 = I_2 = I_3 = 2$.